Spectral analysis

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Motivation or aim of this lecture

The periodogram plays an important role in stationary discrete L_2-processes because $I(\lambda_j)$ can be expressed in terms of the sample covariance function of the process.

Bachmna, Narici and Bechenstein: Fourier and wavelet analysis, page 394, exercise 4.

Spectral Density

Definition (Spectral density)

Let $\{X_t\}$ be a zero-mean stationary time series with autocovariance function $\gamma(\cdot)$ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$. The spectral density of $\{X_t\}$ is the function $f(\cdot)$ defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h), \qquad -\infty < \lambda < \infty,$$

where $e^{i\lambda} = \cos(\lambda) + i\sin(\lambda)$ and $i = \sqrt{-1}$.

Remark: The summability of $|\gamma(\cdot)|$ implies absolute convergence.

The spectral density has the following properties:

1. f is even, i.e., $f(\lambda) = f(-\lambda)$,
2. $f(\lambda) \geq 0 \ \forall \lambda \in (-\pi, \pi]$,
3. $\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda) d\lambda$.

Spectral Density (ii)

Definition (Spectral density (ii))

A function f is the spectral density of a stationary time series $\{X_t\}$ with ACVF $\gamma(\cdot)$ if

1. $f(\lambda) \geq 0 \quad \forall \lambda \in (0, \pi]$,
2. $\gamma = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda$

for all integers h.

Remark: The spectral densities are unique.

Theorem

A real-valued function f defined on $(-\pi, \pi]$ is the spectral density of a stationary process if and only if

1. $f(\lambda) = f(-\lambda)$,
2. $f(\lambda) \geq 0$,
3. $\int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda < \infty$.

Spectral Density (iii)

Corollary

An absolutely summable function $\gamma(\cdot)$ *is the autocovariance function of a stationary time series if and only if it is even and*

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h) > 0, \quad \forall \lambda \in (-\pi, \pi]$$

Spectral Density (iv)

Example

When is

$$\kappa(h) = \begin{cases} 1, & \text{if } h = 0, \\ \rho, & \text{if } h = \pm 1, \\ 0, & \text{otherwise}, \end{cases}$$

the ACVF of a stationary time series?

We note that $\kappa(\cdot)$ is even and nonzero only at lags $0, \pm 1$, then it follows from the corollary that κ is an ACVF if and only if the function

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) = \frac{1}{2\pi}(1 + 2\rho \cos(\lambda))$$

is non negative for all $\lambda \in (-\pi, \pi)$. This is the case if and only if $|\rho| \leq \frac{1}{2}$.

Spectral representation

Example

The time series $X_t = A\cos(\omega t) + B\sin(\omega t)$, where A and B are uncorrelated random variables with mean 0 and variance 1, has ACVF $\gamma(h) = \cos(\omega h)$ ($\Rightarrow$ exercise). This cannot be expressed as $\int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda$, with f a function on $(-\pi, \pi]$.
Still, γ can be written as the Fourier transform of the following discrete distribution function

$$F(\lambda) = \begin{cases} 0 & \text{if } \lambda < -\omega, \\ 0.5 & \text{if } -\omega \leq \lambda \leq \omega, \\ 1.0 & \text{if } \lambda \geq \omega, \end{cases}$$

i.e.,

$$\cos(\omega h) = \int_{(-\pi,\pi]} e^{ih\lambda} dF(\lambda).$$

This is called the spectral representation of the the ACVF.

Spectral representation (ii)

Theorem (Spectral representation of ACVF)

A function $\gamma(\cdot)$ defined on the integers is the ACVF of a stationary rime series if and only if there exists a right-continues, nondecreasing, bounded function F on $[-\pi, \pi]$ with $F(-\pi) = 0$ such that

$$\gamma(h) = \int_{(-\pi,\pi]} e^{ih\lambda} dF(\lambda)$$

for all integers h.

Remark: F is a generalized distribution function on $[-\pi, \pi]$, and is called the spectral distribution function of $\gamma(\cdot)$. The spectrum can be both continuous and discrete.

Linear combination of time series

Exercise 4.5: If $\{X_t\}$ and $\{Y_t\}$ are uncorrelated stationary processes with autocovariance functions $\gamma_X(\cdot)$ and $\gamma_Y(\cdot)$ and spectral distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, respectively, show that the process $\{Z_t = X_t + Y_t\}$ is stationary with autocovariance function $\gamma_Z = \gamma_X + \gamma_Y$ and spectral distribution function $F_Z = F_X + F_Y$.

Spectral representation (iii)

Example (Linear combination of sinusoids)

Consider the process

$$X_t = \sum_{j=1}^{k} (A_j \cos(\omega_j t) + B_j \sin(\omega_j t)), \quad 0 < \omega_1 < \cdots < \omega_k < \pi,$$

where $A_1, B_1, \ldots, A_k, B_k$ are uncrrelated random variables with $E(A_j) = 0$ and $\mathrm{Var}(A_j) = \mathrm{Var}(B_j) = \sigma_j^2,\ j = 1, \ldots, k$. The ACVF of this time series is $\gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(\omega_j h)$ and its spectral distribution function is $F(\lambda) = \sum_{j=1}^{k} \sigma_j^2 F_j(\lambda)$, where

$$F(\lambda) = \begin{cases} 0 & \text{if } \lambda < -\omega_j, \\ 0.5 & \text{if } -\omega_j \le \lambda \le \omega_j, \\ 1.0 & \text{if } \lambda \ge \omega_j, \end{cases}$$

Spectral representation (iv)

Every zero-mean stationary process can be represented as

$$X_t = \int_{(-\pi,\pi]} e^{ih\lambda} dZ(\lambda),$$

where $\{Z(\lambda),\ -\pi < \lambda \leq \pi\}$ is a complex valued process with orthogonal (or uncorrelated) increments. This is the spectral representation of the process.

We do not treat the technical aspects of stochastic integration. It can be shown that a large jump in the spectral distribution function (or a large peak in the spectral density) at frequency $(\pm\omega)$ indicates the presence in the time series of strong sinusoidal components at (or near) ω.

Spectral representation (v)

Example (White noise)

Exercise If $\{X_t\} \sim WN(0, \sigma^2)$, then $\gamma(0) = \sigma^2$ and $\gamma(h) = 0$ for all $|h| > 0$. Show that this process has a flat spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi}, \quad -\pi \leq \lambda \leq \pi.$$

Spectral representation (vi)

Example (AR(1))

Spectral representation (vii)

Example (MA(1))

Similar handwork as for AR(1)

The discrete Fourier transform

Consider the vector $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$. Let

$$\omega_k = \frac{2\pi k}{n}, \quad k = -\left\lfloor \frac{n-1}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

The set of these values is nominated F_n, and is referred to as the Fourier frequencies associated with the sample size n.
We also introduce n vectors

$$e_k = \frac{1}{\sqrt{n}}(e^{i\omega_k}, e^{2i\omega_k}, \dots, e^{ni\omega_k})^T, \quad k = -\left\lfloor \frac{n-1}{2} \right\rfloor, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

The vectors $e_1, e_2, \dots, e_n$ are ortonormal, thus they form a basis for $\mathbb{C}^n$.

The discrete Fourier transform (ii)

As $e_1, e_2, \ldots, e_n$ form a basis for $\mathbb{C}^n$ any $x \in \mathbb{C}^n$ can be expressed as

$$x = \sum_{k=-\left\lfloor \frac{n-1}{2} \right\rfloor}^{\left\lfloor \frac{n}{2} \right\rfloor} a_k e_k,$$

where the coefficients are given by

$$a_k = \frac{1}{n} \sum_{t=1}^{n} x_t e^{-it\omega_k}.$$

The sequence $\{a_k\}$ is called the discrete Fourier transform of the sequence $\{x_1, \ldots, x_n\}$

The periodogram

The periodogram of x is the function

$$I_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^{n} x_t e^{-it\lambda} \right|^2.$$

If λ is one of the Fourier frequencies ω_k, then $I_n(\omega_k) = |a_k|^2$, and it can be shown that

$$|x|^2 = \sum_{k=-\left\lfloor \frac{n-1}{2} \right\rfloor}^{\left\lfloor \frac{n}{2} \right\rfloor} I_n(\omega_k).$$

The value of the periodogram at frequency ω_k is thus the contribution to this sum of squares from the "frequency ω" term $a_k e_k$.

The periodogram (ii)

Theorem

If $x_1, \ldots, x_n$ are any real numbers and ω_k is any of the nonzero Fourier frequencies $2\pi k/n$ in $(-\pi, \pi]$, then

$$I_n(\omega_k) = \sum_{|h|<n} \hat{\gamma}(h) e^{-ih\omega_k},$$

where $\hat{\gamma}$ is the sample ACVF of $x_1, \ldots, x_n$.

Proof.

Since $\sum_{t=1}^{n} e^{-it\omega_k} = 0$ if $\omega \neq 0$, we can subtract the sample mean $\overline{x}$ from x_t in the defining equation of the periodogram. Hence

$$\begin{aligned} I_n(\omega_k) &= n^{-1} \sum_{s=1}^{n} \sum_{t=1}^{n} (x_s - \hat{x})(x_t - \hat{x}) e^{-i(s-t)\omega_k} \\ &= \sum_{|h|<n} \hat{\gamma}(h) e^{-ih\omega_k}. \end{aligned}$$

The periodogram (iii)

Recalling that if $\lambda \in (-\pi, \pi]$ the spectral density is defined as

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h) > 0$$

we see that a natural estimate of the spectral density $f(\lambda)$ is $I_n(\lambda)/(2\pi)$.
This is valid for a large class of stationary time series, but it is not a consistent estimator.

A consistent estimator for the spectral density

A discrete spectral average estimator of the spectral density $f(\lambda)$ has the form

$$\hat{f}(\lambda) = \frac{1}{2\pi} \sum_{|j| \le m_n} W_n(j) I_n(g(n,\lambda) + 2\pi j/n),$$

where the bandwidths m_n satisfy

$$m_n \to \infty \text{ and } m_n/n \to 0 \text{ as } n \to \infty,$$

and the weight functions $W_n(\cdot)$ satisfy

$$\begin{aligned} & W_n(j) = W_n(-j),\ \ W_n(j) \ge 0 \forall j, \\ & \sum_{|j| \le m_n} W_n(j) = 1, \\ & \sum_{|j| \le m_n} W_n^2(j) \to 0 \text{ as } n \to \infty. \end{aligned}$$

A consistent estimator for the spectral density (ii)

Remark: The conditions imposed on the sequences $\{m_n\}$ and $\{W_n(\cdot)\}$ ensure consistency of $\hat{f}(\lambda)$ for $f(\lambda)$ for a very large class of stationary processes including the ARMA processes.
The conditions means that the number of terms in the weighted average goes to infinity as n goes to infinity, and that the mean and variance of $\hat{f}(\lambda)$ converge respectively to $f(\lambda)$ and 0, as n tends to infinity.