# Stationary Processes

Davrondzhon Gafurov

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# Overview

1. Basic Properties and Linear Processes

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## AutoCovariance(ACVF) and AutoCorrelation(ACF)

ACVF: $\gamma(h) = Cov(X_{t+h}, X_t)$

ACF: $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$

$h = 0, \pm 1, \pm 2, ...$

ACVF and ACF provide a useful measure of the degree of dependence among the values of time series at different times.

## Example

Problem: Suppose $X_t$ is a stationary Gaussian* time series and we observed $X_n$. Find a function of $X_n$ that gives us the best predictor of $X_{n+h}$.

Solution: The conditional distribution of $X_{n+h}$ given that $X_n = x_n$ is $N(\mu+\rho(h)(x_n-\mu), \sigma^2(1-\rho(h)^2))$, where $\mu$ and $\sigma^2$ are mean and variance of $X_t$.

$$X_{n+h} = m(X_n) = \mu + \rho(h)(X_n - \mu)$$

The corresponding mean square error is

$$E(X_{n+h} - m(X_n))^2 = \sigma^2(1 - \rho(h)^2)$$

*$X_t$ is a **Gaussian time series** if all of its joint distributiona are multivariate normal, i.e., if for any collection of intergers $i_1, ..., i_n$ the random vector $(X_{i_1}, ..., X_{i_n})'$ has a multivariate normal distribution

## Example(cont')

Remark 1: Prediction of $X_{n+h}$ in terms of $X_n$ is more accurate as $|\rho(h)|$ becomes closer to 1, and as $\rho \to \pm 1$ the best predictor approaches $\mu \pm (X_n - \mu)$ and mean square error approaches 0.

Remark 2: The best linear predictor depends only on the mean and ACf of series $X_t$.

## Properties of ACVF

- $\gamma(0) \geq 0$

- $|\gamma(h)| \leq \gamma(0)$ for all $h$

- $\gamma(h) = \gamma(-h)$ for all $h$

## Properties of ACVF(cont')

Definition: A real-valued function $k$ defined on the integers is **nonnegative definite** if

$$\sum_{i,j=1}^{n} a_i k(i-j) a_j \geq 0$$

for all positive integers $n$ and vectors $\mathbf{a} = (a_1, .., a_n)'$ with real-valued components $a_i$.

Theorem: A real-valued function defined on the integers is the autocovariance function of a stationary time series if and only if it is even and nonnegative definite.

Remark: An autocorrelation function $\rho(\cdot)$ has all properties of an autocovariance function and satisfies the additional condition $\rho(0) = 1$.

## Weakly stationary time series

$X_t$ is a **weakly stationary** if

- $\mu_X(t)$ is independent of t

- $\gamma_X(t+h,t)$ is independent of t for each h

## Strictly stationary time series (SSTS)

$X_t$ is a **strictly stationary time series** if

$$(X_1, X_2, ..., X_n)\prime = (X_{1+h}, X_{2+h}, ..., X_{n+h})\prime$$

, for all integers $h$ and $n \geq 1$

A strictly stationary sequence is one in which the joint distribution of these two vectors (and not just the means and covariances) are the same.

## Properties of SSTS

- The random variables $X_t$ are identically distributed
- $(X_t, X_{t+h})' = (X_1, X_{1+h})'$ for all integers $t$ and $h$
- $X_t$ is weakly stationary if $E(X_t^2) < \infty$ for all $t$
- Weak stationarity does not imply strict stationarity
- An $iid$ sequence is strictly stationary

## Constructing SSTS

One way to construct a time series $X_t$ that is strictly stationary is to "filter" an $iid$ sequence of random variables $Z_t$ and define

$$X_t = g(Z_t, Z_{t-1}, .., Z_{t-q})$$

for some real-valued function $g()$. The $X_t$ is **q-dependent**, i.e. that $X_s$ and $X_t$ are independent whenever $|t - s| < q$. (An $iid$ sequence is 0-dependent). In the same way a stationary time series is **q-correlated** if $\gamma(h) = 0$ whenever $|h| > q$

## The MA(q) Process:

$X_t$ in a **moving-average process of order q** if

$$X_t = Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q},$$

where $Z_t \sim WN(0, \sigma^2)$ and $\theta_1, ..., \theta_q$ are constants

If $X_t$ is a stationary q-correlated time series with mean 0, then it can be represented as the MA(q) process.

## Linear process

The time series $X_t$ in a **linear process** if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

for all $t$, where $Z_t \sim WN(0, \sigma^2)$ and $\psi_j$ is a sequence of constants with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

A linear process is called a **moving average** or **MA($\infty$)** if $\psi_j = 0$ for all $j < 0$, i.e., if

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j},$$

## Introduction to ARMA Processes

The time series $X_t$ in an **ARMA(1,1) process** if it is stationary and satisfies (for every $t$)

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1},$$

where $Z_t \sim WN(0, \sigma^2)$ and $\phi + \theta \neq 0$

## Properties of ARMA

- A stationary solution of the ARMA(1,1) equations exists if and only if $\phi \neq \pm 1$.

- if $|\phi| < 1$ then the unique statioanry solution is given by $X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$. In this case we say that $X_t$ is causal or a casual function of $Z_t$ since $X_t$ can be expressed in terms of the current and past values $Z_s, s \leq t$.

- if $|\phi| > 1$ then the unique statioanry solution is given by $X_t = -\theta\phi^{-1} Z_t - (\phi + \theta) \sum_{j=1}^{\infty} \phi^{-j-1} Z_{t+j}$. The solution is noncasual, since $X_t$ is then a function of $Z_s, s \geq t$.

## Estimation of Mean

$\bar{X}_n = n^{-1}(X_1 + X_2 + ... + X_n)$

If $X_t$ is a stationary time series with mean $\mu$ and $\gamma(\cdot)$, then as $n \to \infty$,

$Var(\bar{X}_n) = E(\bar{X}_n - \mu)^2 \to 0$ if $\mu(n) \to 0$,

$nE(\bar{X}_n - \mu)^2 \to \sum_{|h|} \mu(h)$ if $\sum_{h=-\infty}^{\infty} |\mu(h)| < \infty$

## Estimation of $\mu(\cdot)$ and $\rho(\cdot)$

$$\widehat{\gamma}(h) = n^{-1} \sum_{t=1}^{n-|h|} (X_{t+|h|} - \overline{(}X_n)(X_t - \overline{(}X_n)))$$

and

$$\widehat{\rho}(h) = \frac{\widehat{\gamma}(h)}{\widehat{\gamma}(0)}$$

## Estimation of $\mu(\cdot)$ and $\rho(\cdot)$ (cont')

The sample ACVF has the desirable property that for each $k \geq 1$ the k-dimensional sample covariance matrix

$$\hat{\Gamma}_k = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \ldots & \hat{\gamma}(k-1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) & \ldots & \hat{\gamma}(k-1) \\ . & . & \ldots & . \\ . & . & \ldots & . \\ \hat{\gamma}(k-1) & \hat{\gamma}(k-2) & \ldots & \hat{\gamma}(0) \end{bmatrix}$$

is nonnegative definite.

## Forecasting stationary time series

Predicting the values $X_{n+h}, h > 0$, of a stationary time series with known mean $\mu$ and autocovariance $\gamma$ in terms of values $X_n, ..., X_1$, upto time $n$.

$$P_n X_{n+h} = a_0 + a_1 X_n + a_n X_1$$

Needed to determine the coefficients $a_0, a_1, ..., a_n$, by finding the values that minimizes

$$S(a_0, .., a_n) = E(X_{n+h} - a_0 - a_1 X_n - ... - a_n X_1)^2$$

## Forecasting stationary time series (cont')

$$\frac{\partial S(a_0, ..., a_n)}{\partial a_j} = 0, j = 0, ..., n$$

Evaluation of derivatives gives the equivalent equations

$$E[X_{n+h} - a_0 - \sum_{i=1}^{n} a_i X_{n+1-i}] = 0,$$

$$E[(X_{n+h} - a_0 - \sum_{i=1}^{n} a_i X_{n+1-i}) X_{n+1-j}] = 0, j = 1, ..., n$$

Remark: These 2 last equations determine $P_n X_{n+h}$ uniquely

## Forecasting stationary time series(cont')

$$a_0 = \mu(1 - \sum_{i=1}^{n} a_i)$$

and

$$\Gamma_n \mathbf{a}_n = \gamma_n(h) \tag{1}$$

where
$\mathbf{a}_n = (a_1, ..., a_n)'$,
$\Gamma_n = [\gamma(i-j)]_{i,j=1}^{n}$,
and $\gamma_n(h) = (\gamma(h), \gamma(h+1), ..., \gamma(h+n-1))'$

## Forecasting stationary time series (cont')

$$P_n X_{n+h} = \mu + \sum_{i=1}^{n} a_i (X_{n+1-i} - \mu) \tag{2}$$

The mean square prediction error is

$$E(X_{n+h} - P_n X_{n+h})^2 = \gamma(0) - 2 \sum_{i=1}^{n} a_i \gamma(h+i-1) + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \gamma(i-j) a_j =$$

$$\gamma(0) - \mathbf{a}'_n \gamma_n(h)$$

## Example

Consider a stationary time series defined by

$$X_t = \phi(X_{t-1} + Z_t)$$

where $|\phi| < 1$ and $Z_t \sim WN(0, \sigma^2)$

From formulas 1 and 2

$$P_n X_{n-1} = \mathbf{a}_{n'} \mathbf{X}_n,$$

where $\mathbf{X}_n = (X_n, .., X_1)'$ and

$$\begin{bmatrix} 1 & \phi & \phi^2 & ... & \phi^{n-1} \\ \phi & 1 & \phi & ... & \phi^{n-2} \\ . & . & . & . & . \\ . & . & . & . & . \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} & ... & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix} = \begin{bmatrix} \phi \\ \phi^2 \\ . \\ . \\ \phi^n \end{bmatrix} (*)$$

## Example (cont')

A solution of the equation (*) is

$$\mathbf{a}_n = (\phi, 0, ..., 0)'$$

Mean square error is

$$E(X_{n+h} - P_n X_{n+h})^2 = \sigma^2$$

## Exercises

1. Suppose that $X_1, X_2, ...,$ is a stationary time series with mean $\mu$ and ACF $\rho(\cdot)$. Show that the best predictor of $X_{n+h}$ of the form $aX_n + b$ is obtained by choosing $a = \rho(h)$ and $b = \mu(1 - \rho(h))$.

2. Show that the process $X_t = Acos(\omega t) + Bsin(\omega t), t = 0, \pm 1, ...$ (where A and B are uncorrelated random variables with mean 0 and variance 1 and $\omega$ is a fixed frequency in the interval [0,$\pi$]), is stationary and find its mean and autocovariance function.

3. Find the ACVF of the time series $X_t = Z_t + 0.3Z_{t-1} - 0.4Z_{t-2}$, where $Z_t \sim WN(0, 1)$